MATH 303 – Measures and Integration
Homework 10

Please upload a pdf of your solutions by 23:59 on Monday, December 2. The assignment will be graded out of 16 points (8 for each problem). One problem will be checked for completeness and the other will be graded on correctness and quality. More details on grading, as well as guidelines for mathematical writing, can be found on Moodle.

Problem 1. Let $(X, \mathcal{B}, \mu)$ be a measure space. Suppose $f : X \to \mathbb{C}$ is a measurable function, and $f \in L^{p_0}(\mu)$ for some $p_0 \in [1, \infty)$ and $f \in L^\infty(\mu)$. Show that $\|f\|_\infty = \lim_{p\to\infty} \|f\|_p$.

Solution: If $\|f\|_\infty = 0$, then $f = 0$ a.e., so $\|f\|_p = 0$ for every $p \in [1, \infty)$, and we are done. Suppose $\|f\|_\infty > 0$. We will show

$$\limsup_{p\to\infty} \|f\|_p \le \|f\|_\infty \le \liminf_{p\to\infty} \|f\|_p .$$

Let $\varepsilon \in (0, \|f\|_\infty)$, and consider the set $E = \{|f| > \|f\|_\infty - \varepsilon\}$. Then

$$\int_X |f|^p \, d\mu \ge \int_E (\|f\|_\infty - \varepsilon)^p \, d\mu = (\|f\|_\infty - \varepsilon)^p \mu(E).$$

Taking $p = p_0$, we see that $\mu(E) < \infty$. Moreover, since $\varepsilon > 0$, we have $\mu(E) > 0$ by the definition of the L^∞ norm. Therefore, $\mu(E)^t \to 1$ as $t \to 0$, so

$$\liminf_{p\to\infty} \|f\|_p \ge \lim_{p\to\infty} (\|f\|_\infty - \varepsilon)\mu(E)^{1/p} = \|f\|_\infty - \varepsilon.$$

Letting $\varepsilon \to 0$, we get $\liminf_{p\to\infty} \|f\|_p \ge \|f\|_\infty$.

On the other hand, since $|f| \le \|f\|_\infty$ a.e.,

$$\int_X |f|^p \, d\mu = \int_X |f|^{p-p_0} |f|^{p_0} \, d\mu \le \|f\|_\infty^{p-p_0} \int_X |f|^{p_0} \, d\mu = \|f\|_\infty^{p-p_0} \|f\|_{p_0}^{p_0} .$$

Hence, taking pth roots and then taking a limit,

$$\limsup_{p\to\infty} \|f\|_p \le \lim_{p\to\infty} \|f\|_\infty^{1-p_0/p} \|f\|_{p_0}^{p_0/p} = \|f\|_\infty ,$$

where in the last step we have used that $p_0/p \to 0$ and both of the quantities $\|f\|_\infty$ and $\|f\|_{p_0}$ are finite.

Problem 2.

(a) Let $(X, \mathcal{B}, \mu)$ be a probability space. Let $f : X \to \mathbb{C}$ be a measurable function. Show that for any $1 \le p < q \le \infty$, one has

$$\|f\|_p \le \|f\|_q .$$

(b) We denote by $\ell^p(\mathbb{N})$ the L^p space associated to $\mathbb{N}$ with the counting measure. That is, for $1 \le p < \infty$, $\ell^p(\mathbb{N})$ is the space of sequences $\boldsymbol{a} = (a_n)_{n\in\mathbb{N}}$ with $\sum_{n=1}^\infty |a_n|^p < \infty$, and the norm on the space is given by $\|\boldsymbol{a}\|_p = \left(\sum_{n=1}^\infty |a_n|^p\right)^{1/p}$. The space $\ell^\infty(\mathbb{N})$ is the space of bounded sequences with norm $\|\boldsymbol{a}\|_\infty = \sup_{n\in\mathbb{N}} |a_n|$. Show that for any $1 \le p < q \le \infty$, one has

$$\|\boldsymbol{a}\|_p \ge \|\boldsymbol{a}\|_q$$

Solution: (a) If $q = \infty$, then $|f| \le \|f\|_\infty$ a.e., so $\left(\int_X |f|^p \, d\mu\right)^{1/p} \le \left(\int_X \|f\|_\infty^p \, d\mu\right)^{1/p} = \|f\|_\infty$.

Suppose $q < \infty$. The function $x \mapsto x^{q/p}$ is convex on $[0, \infty)$. Therefore, by Jensen's inequality,

$$\int_X |f|^q \, d\mu = \int_X (|f|^p)^{q/p} \, d\mu \ge \left(\int_X |f|^p \, d\mu\right)^{q/p}.$$

Taking the qth root of both sides gives the desired inequality.

(b) Suppose $q = \infty$, and let $c < \|\boldsymbol{a}\|_\infty$ be arbitrary. Since $\|\boldsymbol{a}\|_\infty = \sup_{n\in\mathbb{N}} |a_n|$, let $k \in \mathbb{N}$ such that $|a_k| > c$. Then

$$\|\boldsymbol{a}\|_p = \left(\sum_{n=1}^\infty |a_n|^p\right)^{1/p} \ge (|a_k|^p)^{1/p} = |a_k| > c.$$

Since $c < \|\boldsymbol{a}\|_\infty$ was arbitrary, this proves $\|\boldsymbol{a}\|_p \ge \|\boldsymbol{a}\|_\infty$.

Now suppose $q < \infty$. We give two different methods of proof.

Method 1: Induction. We will show

$$\left(\sum_{n=1}^N |a_n|^q\right)^{1/q} \le \left(\sum_{n=1}^N |a_n|^p\right)^{1/p} \tag{1}$$

for every $N \in \mathbb{N}$. The desired inequality then follows by taking a limit as $N \to \infty$.

For $N = 1$, both sides of (1) are equal to $|a_1|$. Suppose (1) holds for some N. Then

$$\sum_{n=1}^{N+1} |a_n|^q = \sum_{n=1}^{N} |a_n|^q + |a_{N+1}|^q \le \left(\sum_{n=1}^N |a_n|^p\right)^{q/p} + (|a_{N+1}|^p)^{q/p} \overset{(*)}{\le} \left(\sum_{n=1}^{N+1} |a_n|^p\right)^{q/p},$$

and the desired inequality follows by taking qth roots.

To justify the step $(*)$, let us prove a general inequality of which it is a special case: if $t > 1$ and $x, y \ge 0$, then $x^t + y^t \le (x+y)^t$. If $x = 0$, then $x^t + y^t = y^t = (x+y)^t$. Suppose $x > 0$, and let $u = y/x$. Then $x^t + y^t = x^t(1+u^t)$ and $(x+y)^t = x^t(1+u)^t$, so it suffices to show $1 + u^t \le (1+u)^t$. Let $f(u) = (1+u)^t - 1 - u^t$. Then $f(u) = 0$ and its derivative $f'(u) = t(1+u)^{t-1} - tu^{t-1} = t\left((1+u)^{t-1} - u^{t-1}\right)$ is positive, so f is strictly increasing, whence $f(u) > 0$ for $u > 0$. That is, $1 + u^t \le (1+u)^t$ as claimed.

Method 2: Rescaling. If $\|\boldsymbol{a}\|_q = 0$, there is nothing to prove, so assume $\|\boldsymbol{a}\|_q > 0$. Let $\boldsymbol{u} = \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|_q}$. Then $\sum_{n=1}^\infty |u_n|^q = 1$, so $|u_n| \le 1$ for every $n \in \mathbb{N}$. Therefore, $|u_n|^p \ge |u_n|^q$ for $n \in \mathbb{N}$. Summing over n, we have

$$\frac{\|\boldsymbol{a}\|_p}{\|\boldsymbol{a}\|_q} = \|\boldsymbol{u}\|_p = \left(\sum_{n=1}^\infty |u_n|^p\right)^{1/p} \ge \left(\sum_{n=1}^\infty |u_n|^q\right)^{1/p} = 1,$$

so $\|\boldsymbol{a}\|_p \ge \|\boldsymbol{a}\|_q$.